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INCOMPLETE RELAXATION AND FINITE BETA PLASMAS

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I. Introduction

The setting-up phase of toroidal discharges is generally believed to involve complex turbulent processes which cannot be described in detail even by large 3-D MHD codes. However, the interest within the CTR community centers on describing the gross features of the plasma configuration which result from and which are sustained by turbulent processes.

The treatment of plasma relaxation presented in this paper is a natural generalization of the earlier treatment of Taylor[1] and yields predictions of finite beta plasma confinement. The analysis yields predictions for global reversed-field pinch (RFP) parameters such as β_θ, F, and Θ. In qualitative accordance with the experimental evidence that RFP discharges have a cool outer region of high resistivity, we present a plasma model which permits vanishing wall values of pressure and current density.

II. The Green's Function

We consider the magnetic vector potential in the Coulomb gauge for a simply-connected (spheromak) or a multiple-connected (toroid, infinite straight cylinder) domain bounded by a perfect conductor. The vector potential may be decomposed into two solenoidal parts $\vec{A} = \vec{A}^\Phi + \vec{A}^J$. The curl of $\vec{A}^\Phi$ yields a vacuum magnetic field bearing the constrained magnetic fluxes. $\vec{A}^J$ arises solely from the presence of volume current density $\vec{j}$ within the plasma and satisfies $\hat{n} \times \vec{A}^J|_{b'dy} = 0$, which ensures that $\vec{B} \cdot \hat{n}|_{b'dy} = 0$. Note that $\vec{A}^\Phi = 0$ for simply-connected geometries. Our procedure is the vector analog of the procedure used to derive the electrostatic scalar potential of a charge distribution in a domain bounded by a perfect conductor. Indeed, one may note that the magnetic counterpart of the electrostatic energy is $\epsilon_M = \frac{1}{2c} \int \vec{A}^J \cdot \vec{J} d^3r$ after subtracting off the constant energy of the flux-determined vacuum magnetic fields. Note the lack of explicit dependence on the surface current density.

As in the electrostatic case, a Green's function may be defined. For the case of a domain having a straight cylindrical conducting boundary with circular cross section of radius r_o containing a magnetic flux $\pi r_o^2 B_o$ and current density $\vec{j}(\vec{r})$

$$\vec{A}^{\Phi}(\vec{r}) = \frac{1}{2} B_o\, r\, \hat{\theta} \quad ,$$

$$A_i^J(\vec{r}) = \frac{1}{c} \int d^3r' \; G_{ik}(\vec{r},\vec{r}')\, j_k(\vec{r}') \quad , \tag{1}$$

where $[\nabla^2 \; \overleftrightarrow{G}(\vec{r},\vec{r}')]_{ij} = -4\pi\delta_{ij}\; \delta^{(3)}\,(\vec{r} - \vec{r}')$. If we define the complete set of solenoidal expansion vectors for $\vec{A}^J$ satisfying the boundary condition, orthonormal on $\int \frac{d^3r}{2\pi L}$

$$\vec{\chi}^{m\ell n}(\vec{r}) \equiv \left[\frac{2}{(\gamma_{mn}r_o)^2 - m^2}\right]^{1/2} \frac{1}{|J_m(\gamma_{mn}r_o)|}$$

$$\times \left[\frac{im}{r} J_m(\gamma_{mn}r)\hat{r} - \gamma_{mn}J_m'(\gamma_{mn}r))\hat{\theta}\right] \exp i(m\theta + k_\ell z) \quad , \tag{2a}$$

$$\vec{\Xi}^{m\ell n}(\vec{r}) \equiv \frac{2^{1/2}}{\nu_{mn}\kappa_{m\ell n} r_o |J_m'(\nu_{mn}r_o)|}$$

$$\times \left[ik_\ell \nu_{mn} J_m'(\nu_{mn}r)\;\hat{r} - \frac{mk_\ell}{r}\; J_m(\nu_{mn}r)\hat{\theta} + \nu_{mn}^2 J_m(\nu_{mn}r)\hat{z}\right]$$

$$\times \exp i(m\theta + k_\ell z), \tag{2b}$$

where $J_m'(\gamma_{mn}r_o) = J_m(\nu_{mn}r_o) = 0$, $\kappa_{m\ell n} = (\nu_{mn}^2 + k_\ell^2)^{1/2}$, $\lambda_{m\ell n} = (\gamma_{mn}^2 + k_\ell^2)^{1/2}$, and where $k_\ell = \frac{2\pi\ell}{L}$ (L being the longitudinal period length); then

$$G_{ij}(\vec{r},\vec{r}') = \frac{2}{L} \sum_{m,\ell=-\infty}^{\infty} \; \sum_{n=1}^{\infty} \left[\frac{\chi_i^{m\ell n}(\vec{r})\chi_j^{*m\ell n}(\vec{r}')}{\lambda_{m\ell n}^2} + \frac{\Xi_i^{m\ell n}(\vec{r})\Xi_j^{*m\ell n}(\vec{r}')}{\kappa_{m\ell n}^2}\right] \quad . \tag{3}$$

One should note that $\vec{\chi} \propto [\vec{\nabla}\times\hat{a}\phi(\vec{r})]$ and $\vec{\Xi} \propto \vec{\nabla}\times[\vec{\nabla}\times\hat{a}\Psi(\vec{r})]$ where ϕ and Ψ satisfy scalar Helmholtz equations and $\hat{a} = \hat{z}$. (For a spherical boundary, $\hat{a} = \vec{r}$.) Using Eq. (2), one can verify that the set of Chandrasekhar-Kendall (force-free) eigenvectors used in Ref. 2 is incomplete.

III. Hypothesis of Incomplete Relaxation

Without loss of generality, we shall confine our attention to the case of cylindrical symmetry. (Extension to the helically symmetric case is straightforward.) Equations (1)-(3) yield $A_\theta(r) = \frac{1}{2}rB_o + \sum_{n=1}^{\infty} a_n \, J_1(\gamma_{on}r)$ and $A_z(r) = \sum_{n=1}^{\infty} b_n \, J_o(\nu_{on}r)$ which imply the following complete, orthogonal representation of the magnetic fields: $B_\theta(r) = \sum_{n=1}^{\infty} b_n \, \nu_{on} \, J_1(\nu_{on}r)$ and $B_z(r) = B_o + \sum_{n=1}^{\infty} a_n \, \gamma_{on} \, J_o(\gamma_{on}r)$. The expansion of the magnetic field is in terms of eigenvectors of the resistive diffusion operator for the case of an isotropic homogeneous resistivity η. Therefore, one can associate with each eigenvector a diffusion time scale $t_n = 4\pi\Lambda_n^2/\eta c^2$, where $\Lambda_n = \nu_{on}^{-1}$ or γ_{on}^{-1}.

In a quasi-steady state, we envisage unspecified turbulent processes occurring on a time scale τ which culminate in a dynamo or "α-effect"[3] that opposes resistive diffusion of spectral modes with $t_n \gtrsim \tau$. A direct cascade of magnetic energy through an "inertial" range of the magnetic energy spectrum, due to nonlinear magnetic field-velocity interactions, is thereby resistively dissipated at a Kolmogorov-like scale length $c(\eta\tau/4\pi)^{1/2}$.[4]

The α-effect is then calculable without the customary recourse to kinematic assumptions if one invokes a hypothesis of incomplete relaxation of the plasma: the magnetic energy ε_M of a resistive plasma bounded by a perfect conductor selectively decays with respect to the magnetic helicity K.[5] As a result, the incompletely relaxed state (IRS) of the plasma has the minimum ε_M compatible with constant magnetic flux Φ and constant K, as well as with the resistive truncation of the spectrum.

This hypothesis is implemented by setting

$$a_n = 0, \; n > N_z; \; b_\ell = 0, \; \ell > N_\theta \quad . \tag{4}$$

The additional requirements

$$\frac{\delta(\epsilon_M - \mu K)}{\delta a_n} = 0,\ n < N_z;\ \frac{\delta(\epsilon_M - \mu K)}{\delta b_\ell} = 0,\ \ell < N_\theta \tag{5}$$

determine the remaining coefficients.

The resulting IRS has

1) a non-vanishing β_θ, calculated from

$$\beta_\theta \equiv \frac{16\pi}{r_o^2}\{\int_0^{r_o} r dr[p(r) - p(r_o)]/B_\theta^2(r_o)\},$$

where r_o = wall radius and $\vec{\nabla}p$ is taken to be $\frac{1}{c}\vec{j}\times\vec{B}$,

2) $\vec{j}(r_o) = 0$, and

3) an α-effect calculable from the steady-state condition: $\vec{E} = -\bar{\alpha}(r)\cdot\vec{B}(r) + \eta\vec{J}(r) = 0$.

The global RFP parameters β_θ, F, and Θ describing the IRS are found to be consistent with experimental data; e.g., β_θ is a monotonically increasing function of Θ in the relevant domain. See Figs. 1 and 2.

IV. Discussion

Associated with partial relaxation is a net confinement of plasma energy; i.e., $\beta_\theta > 0$. The associated pressure profile is found to be qualitatively sensitive to the choice of N_θ and N_z. This sensitivity may be attributable to any of the following causes:

1) the presence of localized instabilities that are required to maintain the IRS,
2) the neglect of pressure and velocity fields in Eq. (5),
3) the sharpness of the resistive truncation of Eq. (4), and
4) the neglect of a turbulent current-field interaction, $\frac{1}{c}\delta\vec{J}\times\delta\vec{B}$, whose ensemble average can contribute to $\vec{\nabla}p$.

We have considered various physical mechanisms that lead to smoothed pressure profiles, including a gradual resistive truncation procedure motivated by a suggestion of K. V. Roberts. The dynamics determining the actual resistive truncation is unknown as is also the dynamics governing the approach to the IRS.

Application to helical modes in an RFP is straightforward. The present formulation permits ready evalution of the properties of the IRS for a variety of geometries.[6] We wish to re-emphasize that the theoretical anatomy of an RFP differs from that of a simply-connected spheromak only by the presence of a non-vanishing time-independent $\hat{A}^{\phi}$. Extension of our analysis to cases of anisotropic and inhomogeneous resistivity requires some further analysis.

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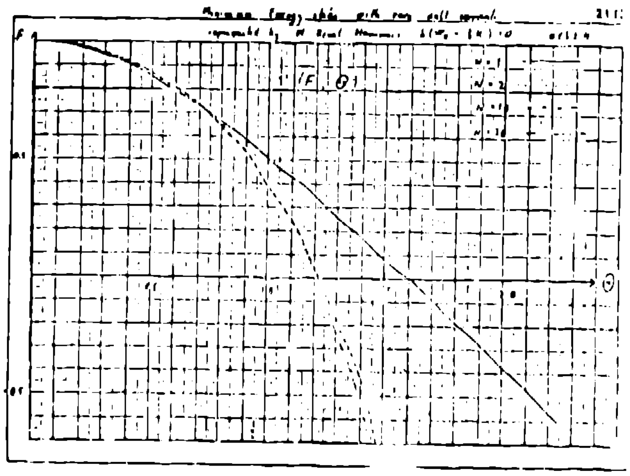

Fig. 1. F as a function of Θ for the cylindrically symmetric minimum energy state. The cases $N=N_{\theta}=N_{z}=1,2,10,20$ are depicted.

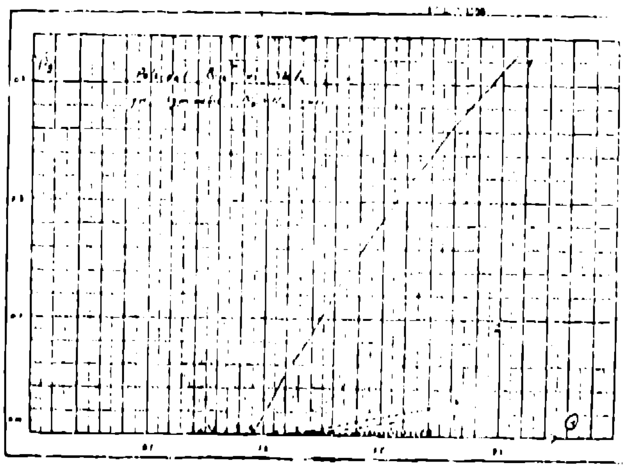

Fig. 2. Poloidal beta as a function of Θ for the cylindrically symmetric minimum energy state. The cases $N_{\theta}=N_{z}=1,2,4,5,10$ are depicted.